

ON THE CENTRALIZERS IN THE WEYL ALGEBRA

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ABSTRACT. Let P, Q be elements of the Weyl algebra W . We prove that if $[Q, P] = 1$, then the centralizer of P is the polynomial algebra $k[P]$.

Introduction

Let k be a characteristic zero field. The Weyl algebra W of index 1 over k is the unital associative k -algebra generated by elements X, Y and the relation $[Y, X] = 1$. This algebra was introduced by Hermann Weyl in order to study the Heisenberg uncertainty principle in quantum mechanics. A detail analysis of W was made in [D]. Among other things, in this paper the author establishes many interesting properties about the centralizer $Z(P)$ of an element P . Another important paper devoted to the investigation of centralizers of elements in the Weyl algebra is [B]. In this note we continue the study of $Z(P)$. Our main result is that if $P, Q \in W$ satisfy $[Q, P] = 1$, then $Z(P) = k[P]$. Dixmier asked in [D] if each endomorphism of W is an automorphism. An affirmative answer immediately implies our theorem, by [D, Th. 9.1].

1 Preliminaries

In this Section we establish some notations and we recall some results from [D]. Let P and Q be non zero elements of W .

Notations 1.1. For $P = \sum a_{ij}X^iY^j$, we write

- $v(P) := \max\{i - j : a_{ij} \neq 0\}$,
- $\ell(P) := \sum_{i-j=v(P)} a_{ij}X^iY^j$,
- $\text{Supp}(P) := \{(i, j) : a_{ij} \neq 0\}$,
- $w(P) := (i_0, i_0 - v(P))$ such that $i_0 = \max\{i : (i, i - v(P)) \in \text{Supp}(\ell(P))\}$,
- $\ell_t(P) := a_{i_0j_0}X^{i_0}Y^{j_0}$, where $(i_0, j_0) = w(P)$,
- $\ell_c(P) := a_{i_0j_0}$, where $(i_0, j_0) = w(P)$,
- $W_+ := \{P \in W : v(P) > 0\}$,
- If $\ell_c(P) = 1$ then we will say that P is *monic*.

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We say that P is aligned with Q and write $P \sim Q$, if

$$km = jl \quad \text{with } w(P) = (k, j) \text{ and } w(Q) = (l, m)$$

Note that \sim is not an equivalence relation (it is so restricted to $\{P : w(P) \neq (0, 0)\}$).

Proposition 1.2. *The following facts hold:*

- (1) $X^k Y^j X^l Y^m = X^{k+l} Y^{j+m} + \sum_{i=1}^{\min(j,l)} i! \binom{j}{i} \binom{l}{i} X^{k+l-i} Y^{j+m-i}$.
- (2) If $P \not\sim Q$, then $[P, Q] \neq 0$ and $w([P, Q]) = w(P) + w(Q) - (1, 1)$.
- (3) $[\ell(P), \ell(Q)] = 0$ implies $P \sim Q$.

Proof. (1) It suffices to prove it when $k = m = 0$. In this case it follows easily by induction on j , using that

$$[Y, X^l] = lX^{l-1} \quad \text{and} \quad [Y^j, X^l] = [Y, X^l]Y^{j-1} + Y[Y^{j-1}, X^l].$$

(2) Let $w(P) = (k, j)$ and $w(Q) = (l, m)$. Since $P \not\sim Q$, it follows from item (1) that

$$\ell_t([\ell_t(P), \ell_t(Q)]) = \left(\binom{j}{1} \binom{l}{1} - \binom{m}{1} \binom{k}{1} \right) \ell_c(P) \ell_c(Q) X^{k+l-1} Y^{j+m-1}.$$

From this fact and item (1) it follows now that

$$\ell_t([P, Q]) = \left(\binom{j}{1} \binom{l}{1} - \binom{m}{1} \binom{k}{1} \right) \ell_c(P) \ell_c(Q) X^{k+l-1} Y^{j+m-1},$$

and so $w([P, Q]) = w(P) + w(Q) - (1, 1)$.

(3) If $P \not\sim Q$, then $\ell(P) \not\sim \ell(Q)$, since $w(\ell(P)) = w(P)$ and $w(\ell(Q)) = w(Q)$. Thus, by item (2) we have $[\ell(P), \ell(Q)] \neq 0$. \square

From item (1) of Proposition 1.2 it follows immediately that

$$\begin{aligned} v(PQ) &= v(P) + v(Q), \\ w(PQ) &= w(P) + w(Q), \\ \ell(PQ) &= \ell(P)\ell(Q), \\ \ell_t(PQ) &= \ell_t(\ell_t(P)\ell_t(Q)), \\ \ell_c(PQ) &= \ell_c(P)\ell_c(Q). \end{aligned}$$

Remark 1.3. Let $\bar{v}(P) := \max\{j - i : a_{ij} \neq 0\}$. All the results in this section admit symmetric versions with $\bar{w}, \bar{\ell}, \bar{\ell}_t, \bar{\ell}_c$ and \bar{W}_+ , defined mimicking the definition of w, ℓ, ℓ_t, ℓ_c and W_+ , using \bar{v} instead of v . In the sequel, when we introduce any symbol denoting an object, the same symbol with an overline will denote the symmetric object.

For each $j \in \mathbb{Z}$ we set

$$W_j := \{P \in W \setminus \{0\} : P = \ell(P) \text{ and } v(P) = j\} \cup \{0\}.$$

Clearly W_j is a subvector space of W and W is a \mathbb{Z} -graded algebra with W_j the homogeneous component of degree j . It is obvious that $\bar{W}_j = W_{-j}$.

2 The structure of the centralizer

For $P \in W$, we let $Z(P)$ denote the centralizer of P . That is, the subalgebra of W consisting of all the Q 's such that $PQ = QP$. The main purpose of this Section is to prove that if there exist $Q \in W$ such that $[Q, P] = 1$, then $Z(P) = k[P]$.

Lemma 2.1. *The following facts hold:*

(1) *For each $j \in \mathbb{N}_0$ and $f \in k[Z]$,*

$$f(XY)X^j = X^j f(XY + j) \quad \text{and} \quad Y^j f(XY) = f(XY + j)Y^j.$$

(2) *For each $j \in \mathbb{Z}$,*

$$W_j = \begin{cases} X^j k[XY] & \text{if } j \geq 0, \\ k[XY]Y^{-j} & \text{if } j < 0. \end{cases}$$

Proof. See [D, 3.2, 3.3]. □

Theorem 2.2. *Let $P \in W_r \setminus k$. If $r = 0$, then $Z(P) = k[XY]$. On the other hand, if $r \neq 0$, then $\dim_k(Z(P) \cap W_j) \leq 1$ for each $j \in \mathbb{Z}$.*

Proof. First assume that $r = 0$. It is evident that $k[XY] \subseteq Z(P)$. In order to check the opposite inclusion, we note that $Z(P)$ is a graded subalgebra of W . Let $Q \in Z(P) \setminus \{0\}$ be an homogeneous element. By item (3) of Proposition 1.2, we have $P \sim Q$ and so $v(Q) = 0$. Now assume that $r > 0$. Write $P = X^r f(XY)$ and take

$$Q \in Z(P) \cap W_j \setminus \{0\}.$$

By item (3) of Proposition 1.2, we know that $P \sim Q$. So $j > 0$ and $Q = X^j g(XY)$ for some $g \in k[Z]$. A direct computation using that

$$f(XY)X^j = X^j f(XY + j) \quad \text{and} \quad g(XY)X^r = X^r g(XY + r)$$

shows that $[P, Q] = 0$ if and only if

$$\frac{f(Z + j)}{f(Z)} = \frac{g(Z + r)}{g(Z)}.$$

Hence, in order to finish the proof in the case $r > 0$, it suffices to check that if $g_1 \neq 0$ satisfies

$$\frac{g_1(Z + r)}{g_1(Z)} = \frac{g(Z + r)}{g(Z)},$$

then $g_1 = \lambda g$ for some $\lambda \in k$. Let $n_0 \in \mathbb{Z}$ be such that $g(n) \neq 0$ and $g_1(n) \neq 0$ for all $n \geq n_0$. Set $\lambda := g_1(n_0)/g(n_0)$. We have

$$g_1(n_0 + r) = g_1(n_0)g(n_0 + r)/g(n_0) = \lambda g(n_0 + r).$$

Iterating the same argument we obtain $g_1(n_0 + ir) = \lambda g(n_0 + ir)$ for all $i > 0$. So the polynomials g_1 and λg coincide. We left the case $r < 0$ to the reader. Use the symmetric version of Proposition 1.2. □

For $P \in W_+$, there exists exactly one pair $(i, j) \in \mathbb{N}_0^2$, with $\gcd(i, j) = 1$, such that $w(P) = (ri, rj)$ for an $r > 0$. We define

$$Z_l(P) = \{Q \in Z(P) \setminus \{0\} : w(Q) = (li, lj)\}$$

for each $l \geq 0$. By item (3) of Proposition 1.2,

$$Z(P) \setminus \{0\} = \bigcup_{l \geq 0} Z_l(P). \tag{2.1}$$

In a similar way we define $\overline{Z}_l(P)$ for every $P \in \overline{W}_+$ and $l \geq 0$.

Theorem 2.3. *For all $P \in W_+$ with $w(P) = (ri, rj)$ as above, there exist elements $R_l \in Z_l(P)$ such that:*

- (1) $Z(P) = \bigoplus_{l \in L} kR_l$, with $L = L(P) \subset \mathbb{N}_0$,
- (2) $\ell_t(R_l) = X^{li}Y^{lj}$,
- (3) $\ell(R_l)\ell(R_h) = \ell(R_{l+h})$.

For $P \in \overline{W}_+$, the symmetric result holds.

Proof. Let $L(P) = \{l \geq 0 : Z_l(P) \neq \emptyset\}$. For all $l \in L(P)$, we fix an $R_l \in Z_l(P)$ with $\ell_t(R_l) = X^{li}Y^{lj}$. Clearly $\bigoplus_{l \in L} kR_l \subseteq Z(P)$. We now prove the opposite inclusion. Let $Q \in Z(P) \setminus \{0\}$ with $w(Q) = (hi, hj)$. Since

$$\ell(Q), \ell(R_h) \in Z(\ell(P)) \cap W_{h(i-j)},$$

by Theorem 2.2 there exists $\lambda_h \in k$ such that $\ell(Q) = \lambda_h \ell(R_h)$. If $Q = \lambda_h R_h$, then $Q \in \bigoplus_{l \in L} kR_l$. Otherwise, $Q - \lambda_h R_h \in Z(P)$ and, by (2.1),

$$w(Q - \lambda_h R_h) = (h'i, h'j),$$

with $0 \leq h' < h$. The same argument yields $\lambda_{h'}$ such that $Q = \lambda_h R_h + \lambda_{h'} R_{h'}$ or

$$Q - \lambda_h R_h - \lambda_{h'} R_{h'} \in Z(P) \quad \text{and} \quad w(Q - \lambda_h R_h - \lambda_{h'} R_{h'}) = (h''i, h''j),$$

with $0 \leq h'' < h'$. Iterating the argument we obtain that $Q \in \bigoplus_{l \in L} kR_l$. So items (1) and (2) hold. Finally, since

$$\ell(R_l)\ell(R_h), \ell(R_{l+h}) \in Z(\ell(P)) \cap W_{(l+h)(i-j)},$$

again, by Theorem 2.2, there exists $\lambda \in k$ such that $\ell(R_l)\ell(R_h) = \lambda \ell(R_{l+h})$. By item (2), necessarily $\lambda = 1$.

When $P \in \overline{W}_+$ the same argument works. \square

Remark 2.4. We assert that $R_0 = 1$. Otherwise $R_0 - 1 \in Z(P) \setminus \{0\}$, and so $R_0 - 1 \sim P$. But this is impossible since $v(P) > 0$ and $v(R_0 - 1) < 0$.

Remark 2.5. Let $P \in W \setminus \{0\}$. If $\overline{v}(P) \leq 0$ and $v(P) \leq 0$, then, necessarily, $\overline{v}(P) = 0 = v(P)$ and $P \in W_0$. In this case $Z(P) = k[XY]$ if $P \notin k$ and $Z(P) = W$ if $P \in k$.

Remark 2.6. Note that in general $R_l R_h \neq R_{l+h}$. So item (1) does not yield a graduation on $Z(P)$. However if $P \in W_+$ is homogeneous (that is $P = \ell(P)$), then we can assume that $R_l = \ell(R_l)$. Hence, by item (3) of the above theorem, $Z(P)$ is graded and, therefore, the map $\varphi : Z(P) \rightarrow k[Z]$, given by $\varphi(R_l) = Z^l$, is a monomorphism of graded algebras. Consequently $Z(P)$ is a monomial algebra. A similar result holds for $P \in \overline{W}_+$, homogeneous.

Lemma 2.7. *Let L be an additive submonoid of \mathbb{N}_0 . Let $r_0 = \min(L \setminus \{0\})$ and $d = \gcd(L)$. For $0 \leq r < r_0$, let $L_r := \{l \in L : l \equiv r \pmod{r_0}\}$. Then*

$$L_r \neq \emptyset \Leftrightarrow d \mid r \quad \text{and} \quad L = \bigcup_{r=0}^{r_0-1} L_r.$$

Proof. First note that the second assertion is trivial and that $L_r \neq \emptyset$ clearly implies that $d \mid r$. We now prove the opposite implication. Let $n_0 := r_0/d$. Since the group generated by L is $d\mathbb{Z}$, there exist $l_0, l_1 \in L$ such that $d = l_1 - l_0$. So

$$l_0 n_0, l_0(n_0 - 1) + l_1, l_0(n_0 - 2) + 2l_1, \dots, l_0 + (n_0 - 1)l_1$$

are elements in the n_0 different L_r with $d \mid r$, as desired. \square

For a fixed $P \in W_+$ we consider $d = \gcd\{l : l \in L(P)\}$ and set $\deg Q = l/d$ for $Q \in Z_l(P)$. Note that

- $\deg(Q_1) > \deg(Q_2)$ if and only if $v(Q_1) > v(Q_2)$,
- for a polynomial $T \in k[S] \setminus k$, where $S \in Z(P)$,

$$\deg T' = \deg T - \deg S,$$

in which T' denotes the usual derivative of the polynomial T .

In the sequel we will use these facts again and again without explicit mention.

For $P \in W_+$, choose an element S_0 of minimal degree in $Z(P) \setminus k$ and set $n_0 := \deg S_0$. For each $0 < l < n_0$, set

$$\widehat{Z}_l(P) := \{R \in Z(P) \setminus \{0\} : \deg(R) \equiv l \pmod{n_0}\}.$$

Since $L(P)$ is an additive submonoid of \mathbb{N}_0 , from Lemma 2.7 it follows that the $\widehat{Z}_l(P)$'s are not empty. Fix $S_l \in \widehat{Z}_l(P)$ of minimal degree.

Corollary 2.8. *We have:*

$$Z(P) = k[S_0] \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1}. \quad (2.2)$$

For $P \in \overline{W}_+$ the symmetric result holds.

Proof. It is clear that

$$k[S_0] \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1} \subseteq Z(P).$$

Consider $R \in Z(P) \setminus \{0\}$. We will prove by induction on $\deg R$ that R is contained in the right side of the equality (2.2). If $\deg R = 0$, then by item (1) of Theorem 2.3 and Remark 2.4, we have $R \in k \subseteq k[S_0]$. Otherwise, there exist $0 \leq r < n_0$ and $l \in \mathbb{N}_0$ such that

$$\deg(S_0^l S_r) = \deg S_r + ln_0 = \deg R.$$

Hence, by Theorem 2.2, we can find $\lambda \in k$ such that $\lambda \ell(S_0^l S_r) = \ell(R)$. Then, by inductive hypothesis, $R - \lambda S_0^l S_r$ belongs to the right side of (2.2), since

$$R - \lambda S_0^l S_r = 0 \quad \text{or} \quad \deg(R - \lambda S_0^l S_r) < \deg R.$$

This finishes the proof. \square

The S_i 's in the previous corollary can be chosen monic (i.e., with $\ell_c(S_i) = 1$), and we will do it so from now on.

By [D, Corollary 4.5] we know that $Z(P)$ is a commutative algebra, and so

$$\partial(T(S)) = T'(S)\partial(S)$$

for any derivation $\partial : Z(P) \rightarrow Z(P)$ and any $T \in k[S]$ with $S \in Z(P)$.

Lemma 2.9. *Let $P \in W_+$ and let*

$$Z(P) = k \oplus k[S_0]S_0 \oplus k[S_0]S_1 \oplus \cdots \oplus k[S_0]S_{n_0-1}$$

be as in Corollary 2.8. Let

$$\partial : Z(P) \rightarrow Z(P)$$

be a derivation and set $J := \{r : \partial(S_r) \neq 0\}$. Then

$$\deg S_r - \deg \partial(S_r) = \deg S_t - \deg \partial(S_t)$$

for all $r, t \in J$.

Proof. Set $g_r := \deg S_r$ and $w_r := \deg \partial(S_r)$. Note that $g_0 = n_0$. Consider the set

$$D := \{w_r - g_r : r \in J\}.$$

We must prove that $\#D = 1$. Assuming that this is not the case will lead us to a contradiction. Take r, t such that

$$w_t - g_t = \max D > w_r - g_r.$$

Since $\deg(S_t^{g_r}) = \deg(S_r^{g_t})$, by Theorem 2.2 we know that there exist $\lambda_0, \lambda_1 \in k$ and $P_i \in k[S_0]$ such that

$$S_t^{g_r} = \lambda_1 S_r^{g_t} + \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i \quad \text{with } P_i S_i = 0 \text{ or } \deg(P_i S_i) < g_t g_r.$$

This implies that

$$U := \deg \partial(S_t^{g_r}) \leq \max\{\deg \partial(S_r^{g_t}), \deg(P_i \partial(S_i)), \deg(\partial(P_i) S_i)\}, \quad (2.3)$$

where we only consider non zero terms $P_i \partial(S_i)$ and $\partial(P_i) S_i$ (note that $\partial(P_i) \neq 0$ only if $\partial(S_0) \neq 0$ and $P_i \notin k$). But (2.3) is impossible since U is strictly greater than each of the terms on the right side. In order to check this, note that

$$\begin{aligned} \partial(S_t^{g_r}) &= g_r S_t^{g_r-1} \partial(S_t), \\ \partial(S_r^{g_t}) &= g_t S_r^{g_t-1} \partial(S_r), \\ \partial(P_i) S_i &= P_i' \partial(S_0) S_i, \end{aligned}$$

and hence,

$$\begin{aligned} \deg \partial(S_t^{g_r}) &= g_r g_t + w_t - g_t = U, \\ \deg \partial(S_r^{g_t}) &= g_r g_t + w_r - g_r < U, \\ \deg(P_i \partial(S_i)) &= \deg P_i + w_i = \deg(P_i S_i) + w_i - g_i < g_t g_r + w_i - g_i \leq U, \\ \deg(\partial(P_i) S_i) &= \deg P_i - g_0 + w_0 + g_i = \deg(P_i S_i) + w_0 - g_0 < U. \end{aligned}$$

This concludes the proof. \square

Proposition 2.10. *Let $P \in W_+$. If $\partial: Z(P) \rightarrow Z(P)$ is a non zero derivation, then $\ker \partial = k$. Moreover, using the same notation as in the previous lemma, if*

$$R = \lambda_0 + \sum_{i=0}^{n_0-1} R_i S_i \in Z(P) \setminus k,$$

then

$$\deg \partial(R) = \deg R + \deg \partial(S_j) - \deg S_j \quad \text{for all } j.$$

Proof. We first prove that

$$\deg \partial(R) \leq \deg R + \deg \partial(S_j) - \deg S_j \quad \text{for all } j \in J. \quad (2.4)$$

By hypothesis $J \neq \emptyset$. As in the proof of Lemma 2.9, we set $g_j := \deg S_j$ and $w_j := \deg \partial(S_j)$ for all $j \in J$. Now fix $j \in J$. Since

$$\deg(\partial(R)) \leq \max\{\deg \partial(R_i S_i) : \partial(R_i S_i) \neq 0\}$$

and

$$\deg(R) = \max\{\deg(R_i S_i) : R_i S_i \neq 0\},$$

it suffices to show that if $\partial(R_i S_i) \neq 0$, then

$$\deg \partial(R_i S_i) \leq \deg(R_i S_i) + w_j - g_j.$$

By Lemma 2.9, if $i \in J$, then

$$\deg(R_i \partial(S_i)) = \deg(R_i S_i) + w_i - g_i = \deg(R_i S_i) + w_j - g_j \quad (2.5)$$

and if $0 \in J$ and $R_i \notin k$, then

$$\begin{aligned} \deg(R_i' \partial(S_0) S_i) &= \deg R_i - g_0 + w_0 + g_i \\ &= \deg(R_i S_i) + w_0 - g_0 \\ &= \deg(R_i S_i) + w_j - g_j. \end{aligned} \quad (2.6)$$

So,

$$\deg \partial(R_i S_i) = \deg(R_i \partial(S_i) + R'_i \partial(S_0) S_i) \leq \deg(R_i S_i) + w_j - g_j,$$

as desired. We now prove that $\ker \partial = k$ or, equivalently, that $J = \{0, \dots, n_0 - 1\}$. Let $0 \leq h < n_0$. As in the proof of Lemma 2.9 we can write

$$S_h^{g_j} = \lambda_1 S_j^{g_h} + \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i \quad \text{with } P_i S_i = 0 \text{ or } \deg(P_i S_i) < g_j g_h. \quad (2.7)$$

Note that $\ell_c(S_h) = \ell_c(S_j) = 1$ and so $\lambda_1 = 1$. Hence, as we have seen above, if $\partial(P_i S_i) \neq 0$, then

$$\deg \partial(P_i S_i) \leq \deg(P_i S_i) + w_j - g_j < g_j g_h + w_j - g_j = \deg \partial(S_j^{g_h}). \quad (2.8)$$

Consequently, since $j \in J$,

$$g_j S_h^{g_j-1} \partial(S_h) = \partial(S_h^{g_j}) = \partial(S_j^{g_h}) + \sum_{i=0}^{n_0-1} \partial(P_i S_i) \neq 0, \quad (2.9)$$

which implies that $h \in J$. It remains to prove that the equality holds in (2.4). For this it will be sufficient to check that for our fixed j and any i ,

$$\deg \partial(R_i S_i) = \deg(R_i S_i) + w_j - g_j. \quad (2.10)$$

In fact, if this is true and

$$\deg \partial(R_h S_h) = \max\{\deg \partial(R_i S_i) : \partial(R_i S_i) \neq 0\},$$

then also

$$\deg(R_h S_h) = \max\{\deg(R_i S_i) : R_i S_i \neq 0\},$$

and so

$$\deg \partial(R) = \deg \partial(R_h S_h) = \deg(R_h S_h) + w_j - g_j = \deg(R) + w_j - g_j.$$

Now we are going to prove (2.10). By (2.5) and (2.6) we are reduced to show that

$$M := \ell(R'_i) \ell(\partial(S_0)) \ell(S_i) + \ell(R_i) \ell(\partial(S_i)) \neq 0 \quad \text{for all } R_i \in Z(P) \setminus k.$$

Write

$$R_i = a_n S_0^n + \dots + a_0 \quad \text{with } a_n \neq 0.$$

By (2.7), we know that $\ell(S_i^{g_0}) = \ell(S_0^{g_i})$ and by (2.8) and (2.9), that

$$g_0 \ell(S_i^{g_0-1}) \ell(\partial(S_i)) = \ell(\partial(S_i^{g_0})) = \ell(\partial(S_0^{g_i})) = g_i \ell(S_0^{g_i-1}) \ell(\partial(S_0)).$$

Hence, in the quotient field of $Z(P)$,

$$\begin{aligned} \ell(\partial(S_i)) &= \frac{g_i}{g_0} \ell(\partial(S_0)) \frac{\ell(S_0)^{g_i-1} \ell(S_i)}{\ell(S_i)^{g_0-1} \ell(S_i)} \\ &= \frac{g_i}{g_0} \ell(\partial(S_0)) \frac{\ell(S_0)^{g_i-1} \ell(S_i)}{\ell(S_0)^{g_i}} \\ &= \frac{g_i}{g_0} \ell(\partial(S_0)) \frac{\ell(S_i)}{\ell(S_0)}, \end{aligned}$$

which implies that

$$\ell(\partial(S_i)) \ell(S_0) = \frac{g_i}{g_0} \ell(\partial(S_0)) \ell(S_i)$$

in $Z(P)$. Therefore, since $S_0 \in W_+$ and $n > 0$,

$$\begin{aligned} M &= \ell(R'_i) \ell(\partial(S_0)) \ell(S_i) + \ell(R_i) \ell(\partial(S_i)) \\ &= n a_n \ell(S_0)^{n-1} \ell(\partial(S_0)) \ell(S_i) + a_n \ell(S_0)^{n-1} \ell(S_0) \ell(\partial(S_i)) \\ &= a_n \ell(S_0)^{n-1} \ell(\partial(S_0)) \ell(S_i) \left(n + \frac{g_i}{g_0} \right) \neq 0, \end{aligned}$$

as desired. \square

Let $P, Q \in W$ such that $[Q, P] = 1$. Since

$$[P, [Q, R]] = [[P, Q], R] + [Q, [P, R]] = [Q, [P, R]],$$

$ad_Q := [Q, -]$ defines a derivation ∂ from $Z(P)$ to $Z(P)$.

Theorem 2.11. *If $P, Q \in W$ satisfy $[Q, P] = 1$, then $Z(P) = k[P]$.*

Proof. Using Lemma 2.1 it is easy to check that if $P \in W_0 \cap \overline{W}_0 = k[XY]$, then there is no Q such that $[Q, P] = 1$. So $P \in W_+ \cup \overline{W}_+$. Assume $P \in W_+$. Let $\partial := ad_Q$ and let S_i, g_i, w_i be as above. Since $g_i > g_0$ for $i > 0$, from Lemma 2.9 it follows that $w_i > w_0$ for $i > 0$. Write

$$P = \lambda_0 + \sum_{i=0}^{n_0-1} P_i S_i.$$

By Proposition 2.10,

$$\deg \partial(P_i S_i) = \deg(P_i S_i) + w_i - g_i = w_i + \deg P_i \equiv w_i \pmod{n_0}$$

for each $P_i \neq 0$. Moreover, by Lemma 2.9

$$w_i \not\equiv w_j \pmod{n_0} \quad \text{for } i \neq j,$$

and so

$$0 = \deg \partial(P) = \max\{w_i + \deg P_i : P_i \neq 0\}.$$

Hence $P_i = 0$ for $i > 0$ since $w_i > w_0 \geq 0$, and $w_0 = 0 = \deg P_0$. Consequently $P = \lambda_0 + \lambda_1 S_0$. We claim that $n_0 = 1$, which concludes the proof. In fact, if $n_0 > 1$, then $\partial(S_l) \in Z(P)$ for $0 < l < n_0$, and so, by Lemma 2.9,

$$\deg \partial(S_l) = w_l = g_l - g_0 \equiv g_l - n_0 \equiv l \pmod{n_0}.$$

Therefore

$$\partial(S_l) \in \widehat{Z}_l(P) \quad \text{and} \quad \deg \partial(S_l) < \deg S_l,$$

which contradicts the minimality of $\deg S_l$. For $P \in \overline{W}_+$, the same argument works, using the symmetric versions of Lemma 2.9 and Proposition 2.10. \square

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